

Geometric U-folds in four dimensions

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We describe a general construction of geometric U-folds compatible with the global formulation of four-dimensional extended supergravity on a differentiable spin manifold. The topology of geometric U-folds depends on certain fiber bundles which encode how supergravity fields are globally glued together. Smooth non-trivial U-folds of this type can exist only in theories where both the scalar and space-time manifolds have non-trivial fundamental group and in addition the configuration of scalar fields of the solution is homotopically non-trivial. Nonetheless, certain geometric U-folds extend to simply-connected backgrounds containing localized sources. Consistency with string theory requires smooth geometric U-folds to be glued using subgroups of the effective discrete U-duality group, implying that the fundamental group of the scalar manifold of such solutions must be a subgroup of the latter. We construct simple examples of geometric U-folds in a generalization of the axion-dilaton model of $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet, whose scalar manifold is a generally non-compact Riemann surface of genus at least two endowed with its uniformizing metric. We also discuss the relation between geometric U-folds and a moduli space of flat connections defined on the scalar manifold, which involves certain character varieties not studied in the literature.

Keywords: Supergravity, Non-geometry, String Geometry, U-folds

I. INTRODUCTION AND MAIN RESULTS

U-folds are consistent backgrounds that incorporate in a non-trivial manner the natural symmetries of a supergravity or string theory [1–7]. In this note, a U-fold means a *supergravity* background which can be obtained by gluing local solutions using U-dualities, aside from local diffeomorphisms and gauge transformations. Such solutions can be promoted to string theory backgrounds only when all U-dualities involved belong to the discrete subgroup allowed by charge quantization.

Many particular constructions of U-folds have been considered in the literature (see, for example, [8–17]), where it was often suggested that some of them do not admit any geometric description. However, no fully general and precise mathematical definition of U-folds has yet been given. Due to this fact, it is unclear which U-folds may admit equivalent (though possibly non-standard) descriptions through ordinary objects of differential geometry, namely objects obtained via constructions involving manifolds and smooth maps satisfying various conditions — the latter of which include fiber bundles. It is thus possible that many backgrounds currently postulated to be “non-geometric” may in fact admit descriptions within the framework of *global* differential geometry — albeit such a description may be “non-standard”.

References [18, 19] considered a particular construction of non-standard geometric solutions based on a

large class of non-simply-connected complex manifolds with properties that are quite different from those of traditional supersymmetric backgrounds. It was argued that those solutions, though constructed geometrically in terms of manifolds and bundles, can be interpreted as U-folds and hence *appear* to be “non-geometric” when viewed from the perspective of patching traditional local solutions using an open cover, in the sense that the gluing of the restrictions of the global solution to the open sets of a cover involves non-trivial U-duality transformations. This shows how it is possible to construct large classes of apparently non-geometric backgrounds using ordinary manifolds and bundles, provided that globally the geometric objects involved are topologically non-trivial.

In this note we propose a general geometric construction of a class of U-folds (called below *geometric U-folds*), which can be considered in any four-dimensional extended supergravity theory. The general global formulation of such theories on a space-time manifold M requires one to specify a certain flat symplectic vector bundle \mathcal{S} defined over the target manifold \mathcal{M} of the kinetic sigma model of scalar fields. The structure group of \mathcal{S} is the U-duality group G_0 of the theory acting in an appropriate symplectic representation ρ which encodes electric-magnetic dualities. More precisely, \mathcal{S} is the vector bundle associated through ρ to a flat principal G -bundle Q defined over \mathcal{M} , where Q must be specified when defining the theory. Given a classical solution Sol of the equations of motion with underlying space-time M , the pull-back of \mathcal{S} through the sigma model map $\Phi : M \rightarrow \mathcal{M}$, which encodes the scalar fields of Sol , gives a flat vector bundle \mathcal{S}_Φ defined over M . When restricted

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to the open sets of a trivializing cover, the transition functions of \mathcal{S}_Φ encode U-duality transformations. The crucial observation is that the bundle \mathcal{S} can be topologically non-trivial, hence \mathcal{S}_Φ can also be non-trivial provided that M and \mathcal{M} are not simply-connected and that the homotopy class of Φ differs from that of the constant map. In this case, at least one of the transition functions of \mathcal{S}_Φ must be non-trivial, so Sol can be interpreted as a non-trivial U-fold. Under the same assumptions on M , \mathcal{M} and Φ , the U-fold interpretation applies even for fluxless solutions (solutions for which all electromagnetic field strengths are identically zero), since in that case Sol can be viewed as a *multivalued* solution of the standard supergravity theory (the theory constructed using the universal cover \mathcal{M}_0 of \mathcal{M}), which is glued from local solutions of the latter using U-duality transformations. Thus a global geometric solution can *appear* to be “non-geometric” when interpreted by patching its restrictions to the open sets of a cover, the reason being, as in [18, 19], that the global solution involves topologically non-trivial geometric objects. As familiar from the cosmic string literature, non-simply-connected spacetimes can often be mimicked by considering source-full solutions defined on simply-connected spacetimes containing localized codimension two sources. In such set-ups, the regular part of the full solution is a source-free solution defined on the complement of all localized sources, a complement which is incomplete and need not be simply connected. In this paper, we consider source-free solutions (which may be restrictions of source-full solutions to the complement of all localized sources).

The standard formulation of four-dimensional supergravity theories involves scalar manifolds \mathcal{M}_0 which are simply-connected; for $\mathcal{N} \geq 3$, these manifolds are symmetric spaces of non-compact type, which are in fact contractible. As a consequence, the flat bundles Q_0 and \mathcal{S}_0 of the standard formulation are always trivial. However, the local computations leading to the construction of supergravity theories only fix the scalar manifold \mathcal{M} up to *local* isometries, i.e. they only determine its universal cover — this cover is the simply-connected manifold \mathcal{M}_0 used in the standard formulation. This observation implies that one can consider generalized supergravity models in which \mathcal{M} is a smooth quotient of \mathcal{M}_0 through the action of a discrete subgroup Γ of the effective¹ U-duality group G_{eff} , in which case $\pi_1(\mathcal{M}) \simeq \Gamma \subset G_{\text{eff}}$. It is such generalized models that admit geometric U-fold solutions. We will see that such U-folds are glued using U-duality transformations belonging to Γ , so they can be lifted to string theory U-folds only when $\pi_1(\mathcal{M})$ is a

subgroup of the discrete U-duality group $G_0(\mathbb{Z}) \subset G_0$ which survives [20] in string theory.

We point out a close relation between geometric U-folds and certain moduli spaces of flat connections defined on the scalar manifold. The latter lead to character varieties that, to our best knowledge, have not been systematically studied in the literature. Finally, we illustrate our construction with two examples. The first is the “generalized axion-dilaton model”, namely $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet with \mathcal{M} given by a (generally non-compact) Riemann surface of genus $g \geq 2$ endowed with its uniformizing metric. The second is $\mathcal{N} = 8$ supergravity with scalar manifold given by a double coset $\Gamma \backslash E_{7(7)} / (\text{SU}(8)/\mathbb{Z}_2)$, where Γ is a discrete subgroup of $E_{7(7)}$. For the generalized axion-dilaton model, we construct explicit geometric U-folds which are similar to those of [21], being sourced by cosmic strings.

The construction outlined in this note leads to various questions for further research. For example, it would be interesting to properly define and study the moduli space of geometric U-folds, which would require studying the relevant character varieties of discrete subgroups of U-duality groups. It would also be interesting to consider similar constructions in $\mathcal{N} = 1$ supergravity and to systematically analyze source-full U-fold solutions. Finally, it would be interesting to construct further explicit examples of geometric U-folds and to study their properties and physics consequences.

II. THE GLOBAL FORMULATION OF EXTENDED FOUR-DIMENSIONAL SUPERGRAVITY THEORIES

In this section we outline the global formulation of ungauged $\mathcal{N} \geq 2$ supergravity² following references [22, 23], describing all non-scalar fields of the theory as global sections of appropriate fiber bundles.

Consider a four-dimensional, oriented, Lorentzian spin manifold (M, g) , whose bundle of complex chiral spinors we denote by S . The *standard* four-dimensional supergravity theories are constructed using certain *simply-connected* Riemannian scalar manifolds $(\mathcal{M}_0, \mathcal{G}_0)$ which are summarized in Table I. The U-duality group G_0 of these theories is summarized in Table II. The Lagrangian contains a total number n of U(1) gauge fields, whose transformation under U-dualities is determined by a certain group morphism $\rho : G_0 \rightarrow \text{Sp}(2n, \mathbb{R})$. We have $n = n_v + m$, where n_v is the number of vector multiplets (which can be non-zero only for $\mathcal{N} \in \{2, 3, 4\}$ while m is

¹ In $\mathcal{N} = 2$ theories, G_{eff} is a discrete quotient of G_0 which acts effectively on the scalar manifold, while for $\mathcal{N} \geq 3$ theories we have $G_{\text{eff}} = G_0$.

² For $\mathcal{N} = 2$, we consider only the theory coupled to vector multiplets.

\mathcal{N}	\mathcal{M}_0	$\dim_{\mathbb{R}} \mathcal{M}_0$	indecomposable
2	PSK	$2n_v$	not necessarily
3	$\frac{\text{SU}(3, n_v)}{\text{S}[\text{U}(3) \times \text{U}(n_v)]}$	$6n_v$	yes
4	$\frac{\text{SL}(2, \mathbb{R})}{\text{U}(1)} \times \frac{\text{SO}_0(6, n_v)}{\text{SO}(6) \times \text{SO}(n_v)}$	$2 + 6n_v$	no
5	$\frac{\text{SU}(1, 5)}{\text{S}[\text{U}(1) \times \text{U}(5)]}$	10	yes
6	$\frac{\text{SO}^*(12)}{\text{U}(6)}$	30	yes
8	$\frac{\text{E}_{7(7)}}{\text{SU}(8)/\mathbb{Z}_2}$	70	yes

TABLE I. The simply-connected scalar manifolds \mathcal{M}_0 of standard four-dimensional supergravity theories, where n_v denotes the number of vector multiplets. For $\mathcal{N} \geq 3$, these scalar manifolds are diffeomorphic with $\mathbb{R}^{\dim \mathcal{M}_0}$ and hence contractible. In the table, $\text{SO}_0(6, n_v)$ denotes the connected component of the identity in the group $\text{SO}(6, n_v)$ (which has two connected components). The abbreviation PSK means a (simply-connected) projective special Kähler manifold.

the number of $\text{U}(1)$ gauge fields in the gravity multiplet. Namely:

1. When $\mathcal{N} = 2$, \mathcal{M}_0 is a simply-connected projective special Kähler (PSK) manifold [24–26] and G_0 is a discrete cover of its group of special isometries³ $\text{Iso}_s(\mathcal{M}_0)$. The dimension of \mathcal{M}_0 (as a real manifold) equals $2n_v$ and we have $n = n_v + 1$, the supplementary $\text{U}(1)$ gauge field being the graviphoton.
2. When $\mathcal{N} \geq 3$, \mathcal{M}_0 is a certain simply-connected globally symmetric space of non-compact type, which is de Rham irreducible except for $\mathcal{N} = 4$ and $n_v \geq 1$, in which case it has two simply-connected and irreducible factors of non-compact type. By the Hadamard-Cartan theorem, it follows that \mathcal{M}_0 is diffeomorphic with $\mathbb{R}^{\dim \mathcal{M}_0}$. Moreover, the U-duality group G_0 is the connected component $\text{Iso}_0(\mathcal{M}_0, \mathcal{G}_0)$ of the identity in the isometry group $\text{Iso}(\mathcal{M}_0, \mathcal{G}_0)$ (see Table II). The U-duality group acts transitively on \mathcal{M}_0 and we have $\mathcal{M}_0 = G_0/H_0$, where $H_0 \subset G_0$ is the isotropy group of this action. For $\mathcal{N} \in \{3, 4\}$, the pure supergravity theory is coupled to n_v vector multiplets and \mathcal{M}_0 is uniquely determined by \mathcal{N} and n_v . For $\mathcal{N} \geq 5$, the theory does not admit coupling to vector multiplets, thus $n_v = 0$, $n = m$ and \mathcal{M}_0 is uniquely determined by \mathcal{N} .

For $\mathcal{N} = 2$ theories, the U-duality group G_0 is a cover of $\text{Iso}_s(\mathcal{M}_0)$ and hence the action of G_0 on \mathcal{M} induced

\mathcal{N}	m	\mathcal{M}_0	H_0	G_0
2	1	PSK	$\text{U}(1)$	cover of $\text{Iso}_s(\mathcal{M}_0)$
3	3	G_0/H_0	$\text{S}[\text{U}(3) \times \text{U}(n_v)]$	$\text{SU}(3, n_v)$
4	6	G_0/H_0	$\text{U}(1) \times \text{SO}(6) \times \text{SO}(n_v)$	$\text{SL}(2, \mathbb{R}) \times \text{SO}_0(6, n_v)$
5	10	G_0/H_0	$\text{S}[\text{U}(1) \times \text{U}(5)]$	$\text{SU}(1, 5)$
6	16	G_0/H_0	$\text{U}(6)$	$\text{SO}^*(12)$
8	56	G_0/H_0	$\text{SU}(8)/\mathbb{Z}_2$	$\text{E}_{7(7)}$

TABLE II. The groups G_0 and H_0 . The symbol n_v denotes the number of vector multiplets while m denotes the total number of $\text{U}(1)$ gauge fields in the gravity multiplet. We have $n = n_v + m$.

through the covering map $G_0 \rightarrow \text{Iso}_s(\mathcal{M})$ may be non-effective (we will see an example of this in Section V). Define the *effective U-duality group* to be the group $G_{\text{eff}} = \text{Iso}_s(\mathcal{M})$ of special isometries, which acts effectively on \mathcal{M}_0 . For $\mathcal{N} \geq 3$ theories, we set $G_{\text{eff}} = G_0$.

In this paper, we will work with the more general choice of scalar manifold:

$$\mathcal{M} = \Gamma \backslash \mathcal{M}_0, \quad (\text{II.1})$$

where $\Gamma \subset G_{\text{eff}}$ is a discrete subgroup of the effective U-duality group such that $\Gamma \backslash \mathcal{M}_0$ is smooth. We endow \mathcal{M} with the metric \mathcal{G} induced from \mathcal{G}_0 and let $\pi : \mathcal{M}_0 \rightarrow \mathcal{M}$ denote the canonical projection. Then $(\mathcal{M}_0, \mathcal{G}_0)$ is the Riemannian universal cover of \mathcal{M} and $\Gamma = \text{Aut}(\pi) \simeq \pi_1(\mathcal{M})$ is the deck group of this cover. Moreover:

1. When $\mathcal{N} = 2$ and Γ is non-trivial, the manifold \mathcal{M} is projective special Kähler.
2. When $\mathcal{N} \geq 3$ and Γ is non-trivial, the manifold \mathcal{M} is a *locally* symmetric space.

Let $\text{Iso}(\mathcal{M}, \mathcal{G})$ denote the isometry group of $(\mathcal{M}, \mathcal{G})$ and K_Γ be the largest subgroup of $\text{Iso}(\mathcal{M}_0, \mathcal{G}_0)$ which contains Γ as its normal subgroup. Then there exists a short exact sequence [28]:

$$1 \longrightarrow \Gamma \longrightarrow K_\Gamma \longrightarrow \text{Iso}(\mathcal{M}, \mathcal{G}) \longrightarrow 1,$$

which can be used to determine the isometry group of $(\mathcal{M}, \mathcal{G})$. The bosonic Lagrangian of the theory based on the scalar manifold (II.1) is globally determined (up to a discrete ambiguity described below) by:

- A principal H_0 -bundle P over \mathcal{M} .
- A flat principal G_0 -bundle Q over \mathcal{M} .

Namely:

1. When $\mathcal{N} = 2$, we have $H_0 = \text{U}(1)$ and P is the canonical circle bundle of \mathcal{M} (the circle bundle of that holomorphic line bundle whose first Chern class equals the Kähler class).

³ Those isometries of \mathcal{M}_0 which preserve the complex structure as well as the flat symplectic connection (sometimes called “duality symmetries” or “duality invariances” [27]).

2. When $\mathcal{N} \geq 3$, H_0 is the isotropy group of the symmetric space \mathcal{M}_0 , while P be the principal H_0 -bundle $\Gamma \backslash G_0 \rightarrow \mathcal{M}$.

When $\Gamma = 1$, the corresponding bundles (which are topologically trivial) are denoted by P_0 and Q_0 and are the bundles used in the standard theory. Notice that Q_0 is trivial as a flat principal bundle since \mathcal{M}_0 is simply-connected, so Q_0 is determined by \mathcal{M}_0 and G_0 up to isomorphism of flat principal bundles. In the general theory (when Γ is non-trivial), the flat connection of Q defines the holonomy representation:

$$\Delta: \Gamma \simeq \pi_1(\mathcal{M}, y) \rightarrow G_0 \quad ,$$

where $y \in \mathcal{M}$ is an arbitrary point. The universal cover \mathcal{M}_0 can be viewed as a principal bundle C over \mathcal{M} with discrete structure group given by Γ . Then Q is isomorphic with the principal G_0 -bundle $C \times_{\Delta} G_0$ associated to C through Δ . Consider the flat vector bundle $\mathcal{S} = Q \times_{\rho} \mathbb{R}^{2n}$ of rank $2n$ over \mathcal{M} , which is associated to Q through the representation ρ . Then \mathcal{S} is isomorphic with the vector bundle $C \times_{\rho \circ \Delta} \mathbb{R}^{2n}$ associated to C through the representation:

$$\rho \circ \Delta: \Gamma \rightarrow \text{Sp}(2n, \mathbb{R}) \quad .$$

The bosonic fields appearing in the Lagrangian are:

- The Lorentzian metric g of M .
- A smooth map $\Phi: M \rightarrow \mathcal{M}$. Using this map we can pull back P , Q and \mathcal{S} to the following bundles defined over M :

$$P_{\Phi} \stackrel{\text{def}}{=} \Phi^*(P) \quad , \quad Q_{\Phi} \stackrel{\text{def}}{=} \Phi^*(Q) \quad , \quad \mathcal{S}_{\Phi} \stackrel{\text{def}}{=} \Phi^*(\mathcal{S}) \quad .$$

- An \mathcal{S}_{Φ} -valued closed 2-form $F \in \Omega_{\text{cl}}^2(M, \mathcal{S}_{\Phi})$, which describes the electric and magnetic field strengths of the $U(1)$ gauge fields.

Notice that Q_{Φ} is a flat principal G_0 -bundle defined over M , whose holonomy representation is given by:

$$\Delta \circ \Phi_*: \pi_1(M, x) \rightarrow G_0 \quad ,$$

where $\Phi_*: \pi_1(M, x) \rightarrow \pi_1(\mathcal{M}, y)$ is the homotopy push-forward through Φ and we took $y = \Phi(x)$ for some $x \in M$. Similarly, $\mathcal{S}_{\Phi} \simeq Q_{\Phi} \times_{\rho} \mathbb{R}^{2n}$ is a flat symplectic vector bundle defined over M , whose holonomy representation is given by $\rho \circ \Delta \circ \Phi_*$. The fermionic field content is determined by two vector bundles:

$$E_G = P_{\Phi} \times_{\theta_G} V_G \quad , \quad E_f = P_{\Phi} \times_{\theta_f} V_f$$

associated to P_{Φ} through complex representations of H_0 , namely the gravitino representation $\theta_G: H_0 \rightarrow \text{Aut}(V_G)$ and the fermionic matter representation $\theta_f: H_0 \rightarrow$

$\text{Aut}(V_f)$, whose precise choice depends on the theory⁴. Here V_G and V_f are complex vector spaces of appropriate dimensions. The fermionic fields in the Lagrangian are:

- The gravitino field, which is a one-form $\Psi \in \Omega^1(M, S \otimes E_G)$ valued in the vector bundle $S \otimes E_G$, where S is the complex spinor bundle of M .
- A spinor $\chi \in \Omega^0(M, S \otimes E_f)$, which is a smooth section of E_f .

The reader may check that the objects introduced above reproduce the standard local (index) formulation of the field content of the theory, including the appropriate local description of the symmetries.

III. GEOMETRIC U-FOLDS

Let Sol be a finite ordered set of fields satisfying the equations of motion of the supergravity theory defined on M . Even though Sol is defined geometrically⁵, it can in certain cases be interpreted as a U-fold when understood by gluing local solutions defined on the sets of an open cover of M . Let $\mathcal{U} \stackrel{\text{def}}{=} \{U_a\}_{a \in I}$ be an open cover of M which is a trivializing cover for both Q_{Φ} and P_{Φ} . Restricting Sol to U_a gives a family $\{\text{Sol}_a\}_{a \in I}$, where:

$$\text{Sol}_a \stackrel{\text{def}}{=} \text{Sol}|_{U_a} \quad , \quad a \in I$$

is a solution of the theory defined on U_a . We are interested in how this family glues to yield the global solution Sol . For intersecting open sets U_a and U_b of the cover, we have two possibilities:

1. The local solutions Sol_a and Sol_b are glued through transformations which do not involve a non-trivial U-duality.
2. The local solutions Sol_a and Sol_b are glued through transformations involving a non-trivial U-duality.

If Q_{Φ} is topologically trivial, then we can arrange that the first case occurs for all pairs of intersecting open sets;

⁴ For certain supergravities, the spinor fields have "Kähler weight" $1/2$, so the representations θ_G and θ_f involve taking the square root of a $U(1)$ sub-bundle R of P_{Φ} which corresponds to R-symmetry. In these cases, fermions should strictly speaking be understood as sections of vector bundles associated to the principal bundle obtained from P_{Φ} upon replacing R with a square root $R^{1/2}$, which exists only when the first Chern class $c_1(R)$ is even and whose choice introduces a discrete ambiguity in the global construction of the theory; see Appendix A.

⁵ It is manifestly described using manifolds and maps of such, including sections of fiber bundles.

in this case, we may in fact find a trivializing cover consisting of the single open set $U = M$. In this situation, we say that Sol is *trivial as a U-fold*. If Q_Φ is topologically non-trivial then the second possibility arises for at least one pair of intersecting open sets of *any* trivializing open cover. In this case, we say that Sol is a *non-trivial geometric U-fold*. Indeed, all fields of the theory, except for the metric and the scalar fields encoded by Φ , are either tensor fields defined on M or global sections of \mathcal{S}_Φ , E_G or E_f . For example, the field-strength $F \in \Omega_{\text{cl}}^2(M, \mathcal{S}_\Phi)$ is a closed two-form taking values in \mathcal{S}_Φ . If Q_Φ is topologically non-trivial then the open cover $\{U_a\}_{a \in I}$ contains at least two intersecting sets U_a , U_b with a non-trivial transition function:

$$g_{ab}: U_a \cap U_b \rightarrow G_0$$

for Q_Φ . Denote by $F_a = F|_{U_a} \in \Omega^2(U_a, \mathcal{S}_\Phi|_{U_a})$ the field-strength of the local solution in U_a and by $F_b = F|_{U_b} \in \Omega^2(U_b, \mathcal{S}_\Phi|_{U_b})$ the field-strength of the local solution in U_b . Then:

$$F_a = (\rho \circ g_{ab})F_b \quad .$$

When Q_Φ is non-trivial, we are thus *forced* to glue F_a to F_b using a non-trivial U-duality transformation for every intersecting pair for which g_{ab} is not identically 1 on $U_a \cap U_b$. Hence:

- A smooth global solution Sol of extended supergravity for which Q_Φ is topologically non-trivial and F is not identically zero is a non-trivial U-fold of the theory based on $(\mathcal{M}, \mathcal{G})$.

Recall that Φ induces a map $\Phi_*: \pi_1(M, x) \rightarrow \pi_1(\mathcal{M}, y)$, where $x \in M$ is such that $\Phi(x) = y$. Even when \mathcal{S}_Φ is trivial or the gauge fields vanish, solutions for which the group $\Phi_*(\pi_1(M, x))$ is non-trivial can be viewed as U-folds of the standard theory based on the scalar manifold $(\mathcal{M}_0, \mathcal{G}_0)$. To see this, let M_0 denote the universal cover of M and $p: M_0 \rightarrow M$ be the canonical projection. For any choice of $x \in M$ and of points $x_0 \in M_0$ and $y_0 \in \mathcal{M}_0$ such that $p(x_0) = x$ and $\pi(y_0) = \Phi(x) = y$, the map Φ lifts to a uniquely-determined map $\Phi_0: M_0 \rightarrow \mathcal{M}_0$ such that $\Phi_0(x_0) = y_0$ and such that the following diagram commutes:

$$\begin{array}{ccc} M_0 & \xrightarrow{\Phi_0} & \mathcal{M}_0 \\ p \downarrow & & \downarrow \pi \\ M & \xrightarrow{\Phi} & \mathcal{M} \end{array}$$

Similarly, all remaining constituent fields of the solution Sol lift to fields defined on M_0 , which together with Φ_0 form a solution Sol₀ (defined on M_0) of the *standard* supergravity theory (which is constructed using $(\mathcal{M}_0, \mathcal{G}_0)$). The original solution Sol of the theory based on $(\mathcal{M}, \mathcal{G})$

can be identified with Sol₀, viewed as a *multivalued* solution of the standard theory defined on M , with monodromies controlled by U-duality transformations belonging to $\Gamma \subset G_{\text{eff}}$. Indeed, let us view the universal cover $\pi: \mathcal{M}_0 \rightarrow \mathcal{M}$ as a principal Γ -bundle C defined over \mathcal{M} . This pulls back through Φ to a principal Γ -bundle $C_\Phi \stackrel{\text{def}}{=} \Phi^*(C)$ defined over M , which in turn pulls back through p to a principal Γ -bundle $C'_\Phi \stackrel{\text{def}}{=} p^*(C_\Phi)$ defined over M_0 . The map Φ_0 can now be viewed as a section $\hat{\Phi}$ of C'_Φ , i.e. as a multivalued global section⁶ of C_Φ . This multivalued section is one-valued (i.e., it descends to an ordinary global section of C_Φ) only when Φ_0 factors through p , i.e. when Φ lifts to a map from M to \mathcal{M}_0 . In turn, this happens iff $\Phi_*(\pi_1(M, x)) = 1$. When $\Phi_*(\pi_1(M, x)) \neq 1$, the multivalued section $\hat{\Phi}$ has monodromy valued in the structure group Γ of C_Φ , which is a subgroup of the effective U-duality group G_{eff} . Thus:

- A smooth global solution Sol of the supergravity theory based on the scalar manifold $(\mathcal{M}, \mathcal{G})$ which has the property that $\Phi_*(\pi_1(M, x)) \neq 1$ is a non-trivial geometric U-fold.

The condition $\Phi_*(\pi_1(M, x)) \neq 1$ requires that both M and \mathcal{M} have non-trivial first homotopy group and that Φ be a homotopically non-trivial map. Similarly, the flat principal bundle Q_Φ (and hence also the flat vector bundle \mathcal{S}_Φ) is trivial unless the holonomy representation $\Delta \circ \Phi_*: \pi_1(M) \rightarrow G_0$ is non-trivial, which requires $\Phi_*(\pi_1(\mathcal{M})) \neq 1$ and that the group morphism Δ be non-trivial (i.e. that Q be a non-trivial flat principal bundle over \mathcal{M}). In particular:

- Every smooth global solution Sol of a standard extended supergravity theory (with simply-connected scalar manifold \mathcal{M}_0) is trivial as a U-fold, as is any solution of the generalized theory based on \mathcal{M} whose underlying space-time is simply-connected or whose scalar field configuration Φ is homotopically trivial.

IV. PRE-CLASSIFYING GEOMETRIC U-FOLDS

Let:

$$\begin{aligned} \mathfrak{M}_{G_0}(\mathcal{M}) &= \text{Hom}(\pi_1(\mathcal{M}), G_0) / G_0 \\ \mathfrak{M}_{G_0}(M) &= \text{Hom}(\pi_1(M), G_0) / G_0 \end{aligned}$$

denote the moduli spaces of flat principal G_0 -bundles over \mathcal{M} and M respectively, where the quotient is

⁶ By definition, a multivalued global section of a fiber bundle $F \rightarrow M$ is an ordinary global section of the bundle $p^*(F) \rightarrow M_0$.

through the adjoint action of G_0 . A smooth map $\Phi : M \rightarrow \mathcal{M}$ is called *admissible* if there exist fields defined on M which, together with Φ , form a solution $\text{Sol} = (\Phi, \dots)$ of the theory defined on M . Let $\mathcal{C}_{\text{ad}}^\infty(M, \mathcal{M})$ denote the space of admissible maps from M to \mathcal{M} . The homotopy space $[M, \mathcal{M}]_{\text{ad}}$ of admissible maps is the space of connected components of $\mathcal{C}_{\text{ad}}^\infty(M, \mathcal{M})$ with respect to the natural topology and can be obtained upon dividing through the homotopy equivalence relation \sim :

$$[M, \mathcal{M}]_{\text{ad}} \stackrel{\text{def}}{=} \pi_0(\mathcal{C}_{\text{ad}}^\infty(M, \mathcal{M})) = \mathcal{C}_{\text{ad}}^\infty(M, \mathcal{M}) / \sim .$$

Since the isomorphism class of $Q_\Phi \stackrel{\text{def}}{=} \Phi^*(Q)$ as a flat bundle depends only on the homotopy class of Φ and on the isomorphism class of Q as a flat bundle, the pull-back operation induces a well-defined map:

$$[M, \mathcal{M}]_{\text{ad}} \times \mathfrak{M}_{G_0}(\mathcal{M}) \ni ([\Phi], [Q]) \rightarrow [\Phi^*(Q)] \in \mathfrak{M}_{G_0}(M) , \quad (\text{IV.1})$$

whose image we denote by $\mathfrak{M}_{G_0}^{\text{ad}}(M)$ and call the *pre-moduli space* of geometric U-folds. The moduli space $\mathfrak{M}(M)$ of geometric U-folds on M (if properly defined) should map to $\mathfrak{M}_{G_0}^{\text{ad}}(M)$ with fiber given by those geometric U-folds which have isomorphic bundles Q_Φ . Effectively describing $\mathfrak{M}(M)$ and $\mathfrak{M}_{G_0}^{\text{ad}}(M)$ appears to be a formidable problem, given the complicated nature of the equations of motion of the theory. Notice that (IV.1) is controlled by the homotopy push-forward:

$$[M, \mathcal{M}]_{\text{ad}} \times \pi_1(M, x) \ni ([\Phi], [\alpha]) \rightarrow \Phi_*([\alpha]) \in \pi_1(\mathcal{M}, y) ,$$

where $\Phi_*([\alpha]) = [\Phi \circ \alpha]$.

V. $\mathcal{N} = 2$ SUPERGRAVITY COUPLED TO ONE VECTOR MULTIPLY

The *standard axion-dilaton model* is $\mathcal{N} = 2$ supergravity coupled to one vector multiplet with simply-connected scalar manifold given by the (open) upper-half plane⁷:

$$\mathcal{M}_0 = \mathbb{H} = \text{PSL}(2, \mathbb{R}) / \text{U}(1) ,$$

equipped with the rescaled Poincaré metric \mathcal{G}_0 of constant Gaussian curvature -2 (scalar curvature -4). The

latter has the squared line element:

$$ds_0^2 = \frac{1}{2(\text{Im}\tau)^2} d\tau d\bar{\tau} = 2(\mathcal{G}_0)_{\tau\bar{\tau}} d\tau d\bar{\tau}$$

and is the unique $\text{PSL}(2, \mathbb{R})$ -invariant metric of the given scalar curvature. The connected component of the isometry group is $\text{Iso}_0(\mathcal{M}_0, \mathcal{G}_0) = \text{PSL}(2, \mathbb{R})$, which is also the group of orientation-preserving isometries. The manifold \mathcal{M}_0 is projective special Kähler⁸ with the following global prepotential defined on the conical special Kähler domain $\mathcal{D} = \{(X^0, X^1) \in \mathbb{C}^2 | \text{Re}(\bar{X}^0 X^1) > 0\}$:

$$\hat{\mathcal{F}}_0(X^0, X^1) = -iX^0 X^1 = -(X^0)^2 \mathcal{F}_0(\tau) ,$$

where $\tau \stackrel{\text{def}}{=} i \frac{X^1}{X^0}$ and $\mathcal{F}_0(\tau) = \tau$. We have $(\mathcal{G}_0)_{\tau\bar{\tau}} = \frac{\partial^2 K_0}{\partial \tau \partial \bar{\tau}} = \frac{1}{(2\text{Im}\tau)^2}$, where K_0 is the (global) Kähler potential in the gauge $X^0 = \frac{i}{2}$:

$$K_0(\tau) = -\ln(\text{Im}\tau) ;$$

notice that we are using the Riemannian (positive-definite) metric on \mathcal{M}_0 . The effective U-duality group is the group of special isometries $G_{\text{eff}} = \text{Iso}_s(\mathcal{M}_0) = \text{PSL}(2, \mathbb{R})$, while the U-duality group is its double cover $G_0 = \text{SL}(2, \mathbb{R})$. The canonical circle bundle is the trivial $\text{U}(1)$ -bundle $P_0 = \mathcal{M}_0 \times \text{U}(1)$. We have $n_v = 1$ and $n = 2$, while the symplectic representation $\rho : G_0 \rightarrow \text{Sp}(4, \mathbb{R})$ is given (up to equivalence of representations) by:

$$\text{SL}(2, \mathbb{R}) \ni A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \rho(A) = \begin{bmatrix} aI_2 & b\Theta_2 \\ c\Theta_2 & dI_2 \end{bmatrix} ,$$

where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Theta_2 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This determines the trivial rank four flat symplectic vector bundle $\mathcal{S}_0 = \mathcal{M}_0 \times \mathbb{R}^4$.

Let $\Gamma \subset G_{\text{eff}} = \text{PSL}(2, \mathbb{R})$ be a Fuchsian group without elliptic elements. The *generalized axion-dilaton model* determined by Γ is $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet with smooth scalar manifold:

$$\mathcal{M} = \Gamma \backslash \mathbb{H} , \quad (\text{V.1})$$

endowed with the constant negative curvature metric \mathcal{G} induced by \mathcal{G}_0 . Thus \mathcal{M} is a (possibly non-compact) smooth Riemann surface of genus $g \geq 2$ while \mathcal{G} is its (rescaled) uniformizing metric (the unique metric on \mathcal{M} which has constant Gaussian curvature equal to -2). By the uniformization theorem, any smooth Riemann surface of genus $g \geq 2$ endowed with its uniformizing

⁷ Since $\text{U}(1)/\mathbb{Z}_2$ is isomorphic with $\text{U}(1)$ through the isogeny $z \rightarrow z^2$, while $\text{SL}(2, \mathbb{R}) \simeq \text{SU}(1, 1)$, we also have the coset presentations $\mathbb{H} \simeq \text{SL}(2, \mathbb{R})/\text{U}(1) \simeq \text{SU}(1, 1)/\text{U}(1)$, which are not minimal since the group appearing in the numerator is a double cover of $\text{Iso}_0(\mathcal{M}_0, \mathcal{G}_0) = \text{PSL}(2, \mathbb{R})$ and $\text{SL}(2, \mathbb{R})$ acts non-effectively on \mathbb{H} .

⁸ Every Riemann surface of genus $g \geq 2$ is projective special Kähler when endowed with its uniformizing metric [29].

metric can be presented as in (V.1). Notice that \mathcal{M} has finite volume when Γ is co-finite and that it is compact when Γ is co-compact. When endowed with the complex structure J induced from \mathbb{H} , the Hermitian manifold $(\mathcal{M}, J, \mathcal{G})$ is projective special Kähler. Any group morphism $\Delta : \Gamma \rightarrow G_0 = \mathrm{SL}(2, \mathbb{R})$ determines a flat principal $\mathrm{SL}(2, \mathbb{R})$ -bundle $Q = C \rtimes_{\Delta} \mathrm{SL}(2, \mathbb{R})$ and a rank four flat symplectic vector bundle $\mathcal{S} = C \rtimes_{\rho \circ \Delta} \mathbb{R}^4$ with monodromy representation $\rho \circ \Delta$. The moduli space of flat $\mathrm{SL}(2, \mathbb{R})$ -bundles on \mathcal{M} is the well-studied character variety:

$$\mathfrak{M}_{\mathrm{SL}(2, \mathbb{R})}(\mathcal{M}) = \mathrm{Hom}(\Gamma, \mathrm{SL}(2, \mathbb{R})) / \mathrm{SL}(2, \mathbb{R}) \quad ,$$

which is closely related to the Teichmüller space of \mathcal{M} . This model admits non-trivial geometric U-folds, which can be promoted to string theory backgrounds only when Γ is a subgroup of $\mathrm{PSL}(2, \mathbb{Z})$. Below, we construct examples of such U-folds.

A. Example: fluxless axion-dilaton U-folds

Consider the generalized axion-dilaton model defined by a Fuchsian group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ without elliptic elements. Focusing on solutions for which the two gauge field strengths vanish identically, we can truncate the bosonic part of the action to:

$$S[g, \tau] = \int_M \left\{ *R + \frac{d\tau \wedge *d\tau}{2(\mathrm{Im}\tau)^2} \right\} \quad . \quad (\mathrm{V}.2)$$

Take the space-time manifold to be of the form:

$$M = \mathbb{R}^2 \times \Sigma \quad ,$$

where Σ is an oriented connected surface without boundary which admits a (possibly-incomplete) flat Riemannian metric g_2 and take g to be a flat Lorentzian metric of the form $g = \eta_2 \times g_2$, where η_2 is the Minkowski metric on \mathbb{R}^2 . Further, assume that τ does not depend on the coordinates of \mathbb{R}^2 , so that it can be viewed as a smooth map $\tau : \Sigma \rightarrow \mathcal{M}$. Then the Einstein equation is satisfied while the equation of motion for τ reduces to:

$$\partial \bar{\partial} \tau + \frac{\partial \tau \bar{\partial} \tau}{\tau \bar{\tau}} = 0 \quad , \quad (\mathrm{V}.3)$$

where ∂ and $\bar{\partial}$ are the Dolbeault differentials defined by the complex structure of Σ corresponding to the conformal class of g_2 . A particular class of solutions of (V.3) is given by maps τ which satisfy $\bar{\partial} \tau = 0$ and hence are holomorphic on Σ . As explained in Section III, these can be viewed as multivalued holomorphic maps $\hat{\tau}$ from Σ to \mathbb{H} whose monodromy representation takes values in Γ and hence involves U-duality transformations. Let $\tau_0 : \Sigma_0 \rightarrow \mathbb{H}$ denote the lift of τ at a point $x_0 \in M_0$,

where Σ_0 is the universal cover of Σ . The universal cover of M is $M_0 = \mathbb{R}^2 \times \Sigma_0$. Distinguish the cases:

1. (Σ, g_2) is complete. Then (Σ, g_2) is an oriented Euclidean space-form and hence must be the Euclidean plane (conformally, the complex plane \mathbb{C}), the flat infinite cylinder (conformally, the complex punctured plane $\mathbb{C} \setminus \{0\}$) or a flat torus (conformally, an elliptic curve). Since \mathcal{M} has genus at least two, the Picard theorem for Riemann surfaces forces τ to be constant, so such solutions are trivial as U-folds.
2. (Σ, g_2) is incomplete. Then Σ can be any open domain of the Euclidean plane. This leads to non-trivial geometric U-folds provided that Σ has non-trivial fundamental group. A physically interesting example is $\Sigma = \mathbb{C} \setminus A$, where $A = \{p_1, \dots, p_k\}$ is a non-empty finite set of points of the complex plane. In this case, the Riemannian universal cover Σ_0 is conformally equivalent with the complex plane or with the Poincaré disk (depending on whether $k = 0, 1$ or $k \geq 2$) and $\hat{\tau}$ can have non-trivial Γ -valued monodromies around each of the points p_j , giving a four-dimensional solution similar to the cosmic string of [21]. The Hodge dual $H_0 \stackrel{\mathrm{def}}{=} *_M d\tau_0 \in \Omega^3(M_0)$ satisfies $dH_0 = 0$ since τ_0 is holomorphic. Thus $H_0 = dB_0$ for a globally-defined two-form $B_0 \in \Omega^2(M_0)$. The standard supergravity theory with scalar manifold $(\mathcal{M}_0, \mathcal{G}_0)$ admits cosmic strings which couple to this potential. Then the solution τ can be interpreted as being sourced by strings with worldvolume $\mathbb{R}^2 \times \{p_j\}$, which are responsible for the monodromies of $\hat{\tau}$.

The supergravity backgrounds constructed above can be lifted to string theory only when Γ is a subgroup of $\mathrm{PSL}(2, \mathbb{Z})$.

Unlike our solutions, the backgrounds of [21] arise in ten-dimensional IIB supergravity. Also notice that the construction of loc. cit. uses the Fuchsian group $\mathrm{PSL}(2, \mathbb{Z})$, which contains elliptic elements. As a consequence, the quotient $\mathrm{PSL}(2, \mathbb{Z}) \backslash \mathbb{H}$ (endowed with the constant negative curvature metric induced by \mathcal{G}_0) is a projective special Kähler *orbifold* (topologically, a once-punctured sphere with two conical orbifold points of orders 2 and 3) which coincides with the moduli space of elliptic curves endowed with its Weil-Petersson metric. This orbifold is in principle not an admissible scalar manifold in supergravity, so the construction of [21] makes physics sense only in non-perturbative IIB string theory, where it gives an F-theory background. By contrast, the construction above works classically in four-dimensional $\mathcal{N} = 2$ supergravity (where it produces solutions containing cosmic string sources), since it involves smooth target manifolds for the scalar field τ .

VI. $\mathcal{N} = 8$ SUPERGRAVITY

The scalar manifold of *standard* $\mathcal{N} = 8$ supergravity is given by:

$$\mathcal{M}_0 = E_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2) \quad .$$

In this case, $H_0 = \mathrm{SU}(8)/\mathbb{Z}_2$ is the maximal compact sub-group of $E_{7(7)}$ and P_0 is the principal H_0 -bundle given by the canonical projection $E_{7(7)} \rightarrow \mathcal{M}_0$. The U-duality group is $G_0 = E_{7(7)}$ and ρ is the 56-dimensional representation of G_0 . The U-duality group has the following polar decomposition [30]:

$$E_{7(7)} \simeq H_0 \times \mathbb{R}^{70} \quad .$$

Since $\pi_1(H_0) = \mathbb{Z}_2$, this gives $\pi_1(E_{7(7)}) = \mathbb{Z}_2$ and shows that \mathcal{M}_0 is diffeomorphic with \mathbb{R}^{70} , hence contractible. Thus every fiber bundle on \mathcal{M}_0 is topologically trivial and the standard theory does not admit non-trivial geometric U-folds.

Let Γ be a discrete subgroup of $E_{7(7)}$. Then any smooth Clifford-Klein form:

$$\mathcal{M} = \Gamma \backslash \mathcal{M}_0 = \Gamma \backslash E_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2)$$

is admissible as a scalar manifold of $\mathcal{N} = 8$ supergravity. At least when \mathcal{M} is non-compact, we expect such generalized models to admit geometric U-fold solutions, which could be promoted to string theory backgrounds when Γ is a subgroup of $E_{7(7)}(\mathbb{Z})$. The pre-moduli space of such geometric U-folds is controlled by the character

variety:

$$\mathfrak{M}_{E_{7(7)}}(\mathcal{M}) = \mathrm{Hom}(\Gamma, E_{7(7)})/E_{7(7)} \quad ,$$

which, to our best knowledge, was not systematically studied in the supergravity literature.

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Appendix A: Square root ambiguity of spinor fields

As mentioned above, for certain supergravities the representations θ_G and θ_f used in the construction of E_G and E_f involve taking the square root of the fundamental representation of a $U(1)$ -sub-bundle of P_Φ , a procedure which generally is obstructed and non-unique. For example, the spinors of $\mathcal{N} = 2$ supergravity are properly-speaking valued in complex vector bundles associated to a square root $P_\Phi^{1/2}$ of the $U(1)$ -bundle P_Φ . Thus $c_1(P_\Phi) = \Phi^* c_1(P)$ must be even. The square roots of P_Φ have first Chern classes lying in the preimage of $c_1(P_\Phi)$ through the endomorphism of $H^2(M, \mathbb{Z})$ given by multiplication with 2. This phenomenon does not seem to have been studied systematically in the supergravity literature.

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